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STABILITY OF EQUILIBRIA OF MECHANICAL SYSTEMS INCLUDING A FLEXIBLE INEXTENSIBLE FILAMENT

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We examine the equilibrium of a mechanical system with a finite number of degrees of freedom, including a flexible inextensible filament, in a stationary potential force field. We investigate the variations of this system's potential energy when the equilibrium position of the generalized coordinates of the system and filament deviates to a new equilibrium position in correspondence with fixed variations of the coordinates mentioned. We have indicated the conditions under which the positive definiteness of the stated variations of potential energy guarantees the stability of the system's equilibrium.

It is assumed that a system S_1 with Lagrange coordinates q $(q_1, \ldots, q_n)$ together with a filament forms a system S_2 admitting of an energy integral and an equilibrium

$$q = 0, \quad x^\circ (q = 0, s), \quad 0 \leqslant s \leqslant l$$

where the functions $x^\circ (q = 0, s)$ express the spatial form of the filament in equilibrium and s is the arc coordinate of the filament's point.

We consider the problem of the stability [1] of such equilibria. In analogy with [2, 3] the variations of the system's potential energy are separated into two terms: $\delta\Pi(q) + \delta\Pi_2$. The term $\delta\Pi(q)$ represents the potential energy variation of system S_2 when passing from the equilibrium to the position $q \neq 0$, $x^\circ (q \neq 0, s)$, where the functions $x^\circ (q \neq 0, s)$ express the form of the filament's equilibrium with respect to a fixed $q \neq 0$. The term $\delta\Pi_2$ expresses the variation of the filament's potential energy when passing from the form $x^\circ (q \neq 0, s)$ to the form $x (q \neq 0, s)$ admissible for the same values of q.

We have formulated sufficient conditions which the functional $\delta\Pi_2$ should satisfy in order that the positive definiteness of the function $\delta\Pi(q)$ would guarantee the stability of the equilibrium. The definition of stability adopted is a natural generalization of Liapunov's definition and ensures the smallness of the deviations of vector q and of all points of the filament from the equilibrium position if the initial values of these deviations and the initial variation of the kinetic energy are sufficiently small. It is shown that functional $\delta\Pi_2$ satisfies the sufficient conditions mentioned in a wide class of equilibria of systems including a homogeneous flexible filament, as well as for certain equilibria of a light filament in an axisymmetric centrifugal field. In two examples of systems of the types indicated all possible stable equilibria are analyzed.

1. Suppose that a mechanical system S_1 has the Lagrange coordinates q $(q_1, \ldots, q_n)$ and that a flexible inextensible filament, included into system S_1, forms a system S_2. The forces and constraints in System S_2 are such that it admits of the energy integral

$$H = T_1 + T_2 + \Pi_1 + \Pi_2$$

where $T(q, q^\cdot)$ is the positive definite kinetic energy relative to $q^\cdot$, $\Pi_1(q)$ is the potential energy of system S_1, T_2 is the filament's kinetic energy, and

$$\Pi_2 = \int_0^l \mu(s)\,\pi(x(s))\,ds$$

is the filament's potentical energy. By s we have designated the distance reckoned along the filament from its point A $(s = 0)$ to a point C $(s \neq 0)$, $0 \leqslant s \leqslant l$. The filament's other end, $s = l$, is denoted by B. We assume that the filament has a linear density $\mu(s)$, $\mu(s)\,\pi(x(s))$ is the potential energy of an element ds situated at a point $x(s)$ of a three-dimensional space, where $x(s)$ is a three-dimensional vector subject to the system's constraints and $\pi(x)$ is a continuous function.

Suppose that system S_2 admits of some equilibrium position $q = q^\cdot = 0$, $x^\circ(q = 0, s)$, $x^{\circ\cdot}(q = 0, s) = 0$ and that for any fixed q from the region $q^2 \leqslant a^2$ the filament admits of a unique equilibrium position $x^\circ(q, s)$ which passes continuously into the equilibrium $x^\circ(0, s)$ as $q^2 \to 0$. Let us consider some admissible displacement of the filament's point C with the arc coordinate $s = s_1$

$$\Delta(q, s_1) = x(q, s_1) - x^\circ(q, s_1)$$

and let us impose on system S_2 the additional constraint $\Delta(q, s_1) = \Delta^\circ$ which signifies that point C has been fixed in space at the point $x(q, s_1) = x^\circ(q, s_1) + \Delta^\circ$.

We assume that the condition $(\Delta^\circ(q, s_1))^2 \leqslant b^2$ is consistent with the system's constraints and delineates in the space of vectors $\Delta(q, s_1)$ a certain closed bounded domain $D(q, s_1, b^2)$, while the union of these domains as the parameter s_1 ranges from zero to l forms a closed bounded domain $D(q, b^2)$. Let us clarify the assumption we have introduced by an example. Suppose that for a fixed q the filament is situated on a surface $\varphi(x, q) = 0$, the filament's ends are fixed at the points A_1 and B_1 of the surface, and the point $C(s_1)$ is found to be a point C_1 on the curve $x^\circ(q, s)$. By $g(D_1, D_2)$ we denote the geodesic distance between points D_1 and D_2 of the surface and we assume that $g(A_1, B_1) < l$. The system's constraints require the relations

$$\varphi(x^\circ(q, s_1) + \Delta^\circ(q, s_1), q) = 0, \quad g(A_1, C_2) \leqslant s_1, \quad g(C_2, B_1) \leqslant l - s_1$$

which, together with the condition $(\Delta^\circ)^2 \leqslant b^2$, delineate the domain $D(q, s_1, b^2)$.

By $x(q, s) = x(\cdot)$ we denote the collection of admissible positions of the filament in the domain $D(q, b^2)$, by the notation $x(\cdot, \Delta^\circ, s_1)$ we select the admissible positions consistent with the additional constraint $\Delta(s_1) = \Delta^\circ$, and by the notation $x^\circ(\cdot, \Delta^\circ, s_1)$ we select the possible equilibria among the curves $x(\cdot, \Delta^\circ, s_1)$. We remark that in this notation the curves $x^\circ(\cdot)$ and $x^\circ(\cdot, 0, s_1)$ coincide for any s_1 as a consequence of our assumption that the equilibrium $x^\circ(\cdot)$ is unique.

By $\delta\Pi_2(x_2(\cdot), x_1(\cdot))$ we denote the variation of the filament's potential energy when passing from curve $x_1(\cdot)$ to curve $x_2(\cdot)$ and we formulate the main assumption concerning the properties of the equilibria $x^\circ(\cdot, \Delta^\circ, s_1)$. Suppose that the equilibrium $x^\circ(\cdot, 0, s_1)$ corresponds to a strict (isolated) minimum

$$\delta\Pi_2(x(\cdot), x^\circ(\cdot, 0, s_1)) > 0 \tag{1.1}$$

and that among the equilibria $x^\circ(\cdot, \Delta^\circ, s_1)$ we can distinguish the equilibria

$X^\circ(\cdot, \Delta^\circ, s_1)$, realizing a minimum (not necessarily strict) among the curves consistent with the additional constraint $\Delta(s_1) = \Delta^\circ$

$$\delta\Pi_2(x(\cdot, \Delta^\circ, s_1), X^\circ(\cdot, \Delta^\circ, s_1)) \geqslant 0 \tag{1.2}$$

and this minimum is continuous in the collection of arguments Δ°, s_1

$$\delta\Pi_2(X^\circ(\cdot, \Delta_1^\circ, s_1), X^\circ(\cdot, \Delta_2^\circ, s_2)) \to 0 \tag{1.3}$$

as

$$|\Delta_1^\circ - \Delta_2^\circ| + |s_2 - s_1| \to 0$$

Lemma 1.1. When conditions (1.1) – (1.3) are fulfilled the functional $\delta\Pi_2$ is positive definite and continuous in the metric

$$\rho_2(x(\cdot), x^\circ(\cdot, 0, s_1)) = \max_{0\leqslant s\leqslant l}(x(\cdot) - x^\circ(\cdot, 0, s_1))^2$$

Proof. We specify a positive number $\varepsilon_2 \leqslant b^2$ and we examine the function

$$P(\Delta^\circ, s_1) = \delta\Pi_2(X^\circ(\cdot, \Delta^\circ, s_1), x^\circ(\cdot, 0, s_1))$$

This quantity is a single-valued function os the variables Δ°, s_1 since it follows from condition (1.3) that even when the variables Δ°, s_1 correspond to several equilibria $X_j^\circ(\cdot, \Delta^\circ, s_1)$, $j = 1, 2, \ldots$, the values of the functional

$$\delta\Pi_2(X_m^\circ(\cdot, \Delta^\circ, s_1), x^\circ(\cdot, 0, s_1)) = \delta\Pi_2(X_k^\circ(\cdot, \Delta^\circ, s_1), x^\circ(\cdot, 0, s_1))$$

for these curves are the same. According to (1.1) the function $P(\Delta^\circ, s_1)$ is positive for $(\Delta^\circ)^2 > 0$, and according to (1.3) it is continuous in the domain $D(q, b^2)$. We subdivide domain D into the domains

$$D_2 = D(q, b^2) \cap [(\Delta^\circ)^2 \leqslant \varepsilon_2], \quad D_3 = D(q, b^2) \setminus D_2$$

and by $\varepsilon_1(\varepsilon_2)$ we denote the minimum of function $P(\Delta^\circ, s_1)$ in domain D_3. We consider some curve $x_1(\cdot)$, by s_1 we denote one of the points of maximum with respect to s of the function $(x_1(q, s) - x^\circ(q, s, 0, s_2))^2$ and we set $\Delta^\circ(s_1) = x_1(q, s_1) - x^\circ(q, s_1, 0, s_2)$. Let function $x_1(\cdot)$ satisfy the estimate

$$\delta\Pi_2(x_1(\cdot), x^\circ(\cdot, 0, s_1)) < \varepsilon_1(\varepsilon_2) \tag{1.4}$$

The estimates $(\Delta^\circ(s_1))^2 < \varepsilon_2$ and $\rho_2(x_1(\cdot), x^\circ(\cdot, 0, s_2)) < \varepsilon_2$ follow from estimates (1.1), (1.2), (1.4). In summary we have shown that for any $\varepsilon_2 > 0$ we can find $\varepsilon_1(\varepsilon_2) > 0$ such that the estimate $\rho_2 < \varepsilon_2$ would follow from estimate (1.4). The latter property coincides with the definition of positive definiteness [4] of the functional $\delta\Pi_2$ in the metric ρ_2. Continuity, i.e. the relation

$$\delta\Pi_2(x_1(\cdot), x^\circ(\cdot, 0, s_2)) \to 0 \quad \text{as} \quad \rho_2(x_1(\cdot), x^\circ(\cdot, 0, s_2)) \to 0$$

serves as a simple corollary of the continuity of the function $\pi(x)$. Lemma 1.1 is proved.

Let us formulate a definition and a theorem which are natural generalizations of Liapunov's definition of stability and of Lagrange's well-known theorem, by introducing the complete metric

$$\rho = q^2 + \dot{q}^2 + T_2 + \rho_2(x(\cdot), x^\circ(\cdot, 0, s_1))$$

and the variation $\delta\Pi(q)$ of the potential energy of system S_2 when passing from the equilibrium $q = 0$, $x^\circ(0, s, 0, s_1)$ to the position

$$q \neq 0, \; x^\circ(q \neq 0, s, 0, s_1)$$

Definition. If for any $\varepsilon_1 > 0$ we can find $\varepsilon_2(\varepsilon_1) > 0$ such that the estimate $\rho(t > 0) < \varepsilon_1$ follows from the estimate $\rho(t = 0) < \varepsilon_2$, then the equilibrium is stable in metric ρ.

Theorem. If $\delta\Pi(q)$ is a positive-definite function of coordinates q, and $\delta\Pi_2$ satisfies properties (1.1) – (1.3), then the equilibrium is stable in metric ρ.

To prove the theorem it is sufficient to note that positive definiteness and continuity of functional $\delta\Pi_2$ in metric ρ follow from Lemma 1.1 and the theorem's hypotheses, and then to apply the results of [4].

2. We consider two problems for a homogeneous ($\mu = \mu_0 = \text{const}$) filament.

Problem 2.1. For a fixed q the filament's ends A $(s = 0)$ and B $(s = l)$ are fixed in space, while the filament itself sags freely under the action of gravity. Having directed the x-axis horizontally to the right and the y-axis vertically upward and locating both axes in a vertical plane containing the points A and B, we obtain $\pi_1 = gy$.

Problem 2.2. For fixed q the filament's ends A and B are fastened to a plane P rotating with constant angular velocity ω around a fixed straight line x, while the filament itself slides freely on plane P. Having directed the y-axis perpendicularly to the x-axis in plane P, we obtain $\pi_2 = -(^1/_2)\omega^2 y^2$.

Denoting the coordinates of points A and B by (x_1, y_1) and (x_2, y_2), respectively, we pose the problem of seeking curves $\overset{\circ}{y}_{(1),(2)}$ which realize the absolute minima of the integrals

$$\Pi_2^{(1),(2)} = \int_{x_1}^{x_2} \pi_{1,2} \sqrt{1 + y_x'^2}\, dx$$

subject to the isoperimetric conditions

$$\int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx = l$$

In order that the desired curves $\overset{\circ}{y}_{(1),(2)}(x)$ corresponding to an absolute minimum lie among the extremal integrals

$$G^{(1),(2)} = \Pi_2^{(1),(2)} + \lambda_{(1),(2)} \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

two additional conditions suffice.

2.3. The curves $\overset{\circ}{y}_{(1),(2)}(x)$ admit of a continuous first derivative in x.

2.4. They have no rectilinear segments.

We assume that $x_2 > x_1$ and we briefly sketch a plan for proving the existence of curve $\overset{\circ}{y}_{(1)}(x)'$ and the fulfillment of properties 2.3, 2.4.

2.5. We replace the filament by a system of homogeneous segments Δ_i, $i = 1, \ldots, p$, joined by ideal hinges and having the same length l / p. After this the functional $\Pi_2^{(1)}$ becomes the function $\Pi_p{}^1(\alpha)$, where α is a vector of dimension p and its components α_i are the angles between the segments Δ_i and the x-axis, constrained by the junction conditions

$$(l / p)(\cos\alpha_1 + \ldots + \cos\alpha_p) + x_1 - x_2 = f_{1,p}(\alpha) = 0$$

$$(l / p)(\sin\alpha_1 + \ldots + \sin\alpha_p) + y_1 - y_2 = f_{2,p}(\alpha) = 0$$

Let us show that the absolute minimum is reached on a strictly convex polygonal line. In fact, if the polygonal line is concave between hinges g_1, g_2, then the symmetric reflection of the polygonal line relative to the middle of segment g_1, g_2 decreases the potential energy. It can also be shown that at the absolute minimum not one of the segments Δ_{i+1} can be a prolongation of segment Δ_i, i.e. the polygonal line is strictly convex and has no vertical segments. The proof is also carried out by contradiction by making an admissible displacement lessening the potential energy.

From what has been presented above it follows that the absolute minimum of the function $\Pi_{2,p}^{(1)}(\alpha)$ lies at a stationary point of the function

$$G_p^{(1)} = \Pi_{2,p}^{(1)}(\alpha) + \nu_1 f_{1,p}(\alpha) + \nu_2 f_{2,p}(\alpha)$$

(ν_1, ν_2 are certain constants). An analysis of the equations of the stationary point shows that there exists only one strictly increasing sequence $\alpha_{1,p}, \ldots, \alpha_{p,p}$ satisfying the stationarity condition. This sequence corresponds to a convex polygonal line. It can be shown that the step function

$$\alpha_p(x) = \alpha_{i,p}, \qquad x \in \left(\sum^{i-1} \cos\alpha_{j,p}, \sum^{i} \cos\alpha_{j,p}\right)$$

expressing the dependence of the polygonal line's angle of inclination on the coordinate x, converges as $p \to \infty$ to some strictly increasing function $\alpha(x)$, continuous and contained within the limits $+\pi/2 > \alpha > -\pi/2$. This completes the proof of properties 2.3, 2.4 for the curve $\overset{\circ}{y}_{(1)}(x)$.

The proof of properties 2.3, 2.4 in Problem 2.2 can be successfully carried out by an analogous scheme, but under the essential assumption that points A and B are not separated by the x-axis in such a way that by a suitable choice of the y-axis we can simultaneously satisfy the estimates $y_2 \geqslant y_1 \geqslant 0$. Under such a choice of the axis we can prove that the function $\overset{\circ}{y}_{(2)}(x)$ is strictly concave ($\alpha(x)$ decreases monotonically). Curve $\overset{\circ}{y}_{(1)}(x)$ is the very well-known catenary; the structure of the curve $\overset{\circ}{y}_{(2)}(x)$ needs detailing.

Let points A and B lie on the x-axis, $A(0, 0)$ and $B(\xi > 0, 0)$, then [5] all stationary curves of functional $G^{(2)}$ are found in the class of elliptic sines

$$y = \sqrt{b_1^2 - a_1^2}\,\mathrm{sn}\,(x\sqrt{2}/a_1 k') \tag{2.1}$$

$$k'^2 = 1 - k^2, \qquad k^2 = (b_1^2 - a_1^2)/(b_1^2 + a_1^2)$$

The quantity $k^2(l/\xi)$ is determined uniquely from the equation

$$\frac{(l-\xi)}{(l+\xi)} = \frac{k^2\int_0^1 (1-t^2)^{1/2}(1-k^2t^2)^{-1/2}\,dt}{\int_0^1 (1-k^2t^2)^{1/2}(1-t^2)^{-1/2}\,dt}$$

The function $t = \mathrm{sn}\,z$ is the inverse of the elliptic integral

$$z = \int_0^t (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}\,dt$$

Note that $k^2(l/\xi)$ increases monotonically from 0 to 1 as l/ξ increases from 1 to ∞. For given l and ξ the constant a_1 can take a countable set of values

$$a_1(m) = \xi\sqrt{2}/2mKk', \qquad m = 1, 2, \ldots$$

$$K = \int_0^1 ((1-t^2)(1-k^2t^2))^{-1/2}\,dt$$

This means that a countable set of equilibria exist. The firat three equilibria are shown in Fig. 1; the second and third halfwaves are obtained from the first by a similarity transformation relative to the point A (0, 0) with coefficients $^1/_2$ and $^1/_3$.

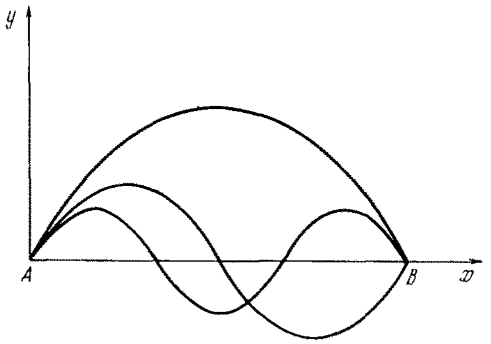

Fig. 1

Geometrically it is obvious that the minimum of the potential energy is reached on the first halfwave. From the preceding it follows that it is absolute minimum; the strictness of this is evident.

Let us show that the second, third, and remaining curves do not realize a minimum at all. To do this we shift the filament from the second position ($m = 2$) to a neighboring admissible one by a similarity transformation of the upper and lower halfwaves relative to points A and B with coefficients $1 + u$ and $1 - u$, where $0 < u < 1$. Denoting the potential energy of the second equilibrium position by $\Pi_2^{(2)}$ ($m = 2$), we can easily verify that the potential energies of the transformed halfwaves take the values $^1/_2\ (1 + u)^3\ \Pi_2^{(2)}(m = 2)$, and $^1/_2\ (1 - u)^3\Pi_2^{(2)}$ $(m = 2)$, while under this transformation the variation $\Delta\Pi_2^{(2)}$ equals $3u^2\Pi_2^{(2)}\ (m = 2) < 0$, since

$$\Pi_2^{(2)}\ (m = 2) < 0$$

2.6. We now show that if points A and B are not separated by the x-axis, then there always exists a unique curve Γ of length l, belonging, to within a shift along the x-axis, to family (2.1), passing through A and B, and having no more than one halfwave on the segment $[x_1,\ x_2]$. Denoting

$$\Delta x = x_2 - x_1 > 0,\ \Delta y = y_2 - y_1 > 0,\ y_1\ /\ \Delta x = \eta,\ \Delta y\ /\ \Delta x = \zeta$$

we examine the family of straight lines

$$y = \zeta x + e, \qquad e > 0 \tag{2.2}$$

depending on a positive parameter e, and a subfamily of family (2.1), depending on the parameter b_1^2, in which the equality $a_1^2 = b_1^2 - 1$ distinguishes the curves with unit amplitude $1 = b_1^2 - a_1^2$.

If the initial (for $x = 0$) value of the derivative on the line of the subfamily indicated satisfies the estimate

$$y_x'\,(x = 0) = \sqrt{2}\,/\,\sqrt{b_1^2 - 1} > \zeta \tag{2.3}$$

then the straight line $y = \zeta x + e$ has two points $(x_1\ (e),\ y_1\ (e))$ and $(x_2\ (e),\ y_2\ (e))$ of intersection with the first halfwave, and the ratio $y_1\ (e)\ /\ (x_2\ (e) - x_1\ (e))$ increases monotonically from zero to infinity as the parameter e increases from zero to the value e_1 for which the straight line $y = \zeta x + e_1$ is tangent to the first halfwave, while the intersection points merge. This signifies that in accordance with estimate (2.3) the equality $\eta = y_1(e)\,/\,[x_2(e) - x_1(e)]'$ is fulfilled for a unique value of e for any $b_1^{(1)}$. A similarity transformation relative to point (0, 0) with coefficient

$$\Delta x\,/\,(x_2\,(e) - x_1\,(e))$$

and a shift along the x-axis lead to a curve Γ_1 on which the junction conditions are fulfilled and which belongs, to within a shift, to family (2.1). By decreasing the quantity $b_1^{(1)}$ to the value $b_1^{(2)}$ (increasing the initial derivative) and repeating the operation, we obtain curve Γ_2, where it is obvious from the construction that the length of the arc $(\breve{AB})_2 = l_2 > l_1 = (\breve{AB})_1$(the length of an arc of curve Γ_1). Thus the ratio $l / \sqrt{\Delta x^2 + \Delta y^2}$ is a strictly monotonic function of the parameter $b^2{}_1$, ranging over the interval $(1, \infty)$ as $b_1{}^2$ varies within the limits $\zeta < \sqrt{2} / \sqrt{b_1{}^2 - 1} < \infty$, i.e. the curve Γ exists and is unique.

2.7. The curve Γ realizes the desired absolute minimum, i.e. is the curve $y_{(2)}{}^\circ(x)$. Let us prolong curve Γ by means of the elliptic sine equation up to intersection with the x-axis at points A' and B'. We obtain a curve Γ' of length l', which, as was shown above, realizes the absolute minimum for a filament of length l' with fastenings at A' and B'. It is obvious that the admissible deformations of curve Γ lie among the admissible deformations of curve Γ'. Consequently, the curve Γ realizes the absolute minimum. In summary, the two assertions can be taken as established by the analyses made.

2.8. For any admissible constraint $\Delta(q, s_1) = \Delta^\circ$, in Problem 2.1 there exists a unique curve $X^\circ(\cdot, \Delta^\circ, s_1)$. This curve consists of two segments of catenaries passing through the point $C_1(x^\circ(q, s) + \Delta^\circ)$ and lying in vertical planes containing the pairs (A, C_1), (C_1, B).

2.9. In Problem 2.2, for the conditions $y_2 > y_1 > 0$ and for a filament situated along the curve $y_{(2)}{}^\circ(x)$, there also exists a unique curve $X^\circ(\cdot, \Delta^\circ, s_1)$ which consists of two incomplete halfwaves of the elliptic sine. The uniqueness of the curve indicated is a consequence of the preceding analyses for all regions $(\Delta^\circ)^2 \leqslant b^2$ not containing points $y < 0$. If one or both points $(y_1 = 0,\ y_2 > 0)$ A and B lie on the x-axis, then the region $(\Delta^\circ)^2 \leqslant b^2$ always contains the displacements of point $C(s_1)$ in the region $y < 0$ for values of s_1 sufficiently close to zero.

Denoting the coordinates of point C_1 by x_3, y_3, we select, for $y_3 \geqslant 0$, a curve $X^{\circ +}$ in the form of two incomplete positive halfwaves, while for $y_3 < 0$ we reflect the point $B(x_2, y_2)$ into the point $B'(x_2, -y_2)$ and we select the curve in the form of two incomplete negative halfwaves, tracing them through the pairs (A, C_1), (C_1, B'). Any curve connecting the points (C_1, B) must intersect the x-axis at least once at a point E. By reflecting the curve (B, E) symmetrically with respect to the x-axis, we obtain the curve (A, E, B'), whose potential energy is obviously not less than the energy of curve $X^{\circ -}$. Thus, property (1.2) is fulfilled for the curves $X^{\circ -}$. Obviously, property (1.3) is also valid. Curve $X^{\circ -}$ is not admitted by junction conditions when $y_2 > 0$ and formally does not belong to the class of X°. However, in the proof of Lemma 1.1, we nowhere made use of the fact that the curves X° are admissible equilibria. The latter requirement is not necessary, although it sharply facilitates the search for curves satisfying conditions (1.2), (1.3).

3. Let us consider a particular case of Problem 2.1, introducing an additional notation. Suppose that a heavy rigid body (a pendulum) can rotate around a horizontal O-axis and that the points A and B of fastening of the filament's ends are located on the rigid body in a vertical plane perpendicular to the axis of rotation. Let O_1 be the middle of segment AB, a be the length of the segment AB, m_1, m_2 be the masses of the body and of the filament, respectively, l be the length and m_2 / l the density of the filament, G be the center of gravity of the body, m_2 be the additional point mass located in the point O_1, and b be the distance GO. From point O_1 we draw a horizontal x-axis to the right and a vertical y-axis upward, and by φ we denote the angle between the direction A, B and the x-axis.

In this coordinate system the junction conditions take the form

$$x_{1,2} = \pm (a / 2) \cos \varphi, \qquad y_{1,2} = \pm (a / 2) \sin \varphi \tag{3.1}$$

If the filament is located on a convex $(\alpha > 0)$ catenary

$$y = \alpha \operatorname{ch} [(x - \beta) / \alpha] - \gamma \tag{3.2}$$

then the quantities α, β, γ are determined as functions of parameters a, l and of angle φ by the junction conditions (3.1) and by the equation for length preservation

$$l = \alpha (\operatorname{sh} [(x_1 - \beta) / \alpha] - \operatorname{sh} [(x_2 - \beta) / \alpha]) \tag{3.3}$$

The sum $\Pi_1 + \Pi_2$ of the potential energies of the body and filament can be represented as the sum $\Pi_3 + \Pi_4$ of the potential energy of the body with an additional mass m_2 and the potential energy of the filament at a fixed point O_1. When computing energy Π_3 we shall examine two different methods of fastening.

3.1. The straight line OG passes through point O_1, perpendicularly to segment AB. Then, assuming that the center of gravity G lies above point O when $\varphi = 0$, we obtain (3.4)

$$\Pi_3^{(1)} = (m_1 + m_2)\, gb \cos \varphi, \quad \Pi_3^{(1)\prime} = -M_3^{(1)} = -(m_1 + m_2) gb \sin \varphi$$

where the derivative $\Pi_3^{(1)\prime}$ equals the moment of the force of gravity relative to point O, taken with opposite sign.

3.2. The straight line OG coincides with the straight line AB. Then, assuming that the center of gravity G lies to the left of the point O when $\varphi = 0$, we obtain

$$\Pi_3^{(2)} = -(m_1 + m_2) gb \sin \varphi, \quad \Pi_3^{(2)\prime} = -M_3^{(2)} = -(m_1 + m_2) gb \cos \varphi \tag{3.5}$$

The term Π_4 and the moment $M_4 = -\Pi_4'$ have the forms

$$\Pi_4 = \frac{m_2 g}{l} \int_{x_1}^{x_2} y \sqrt{1 + y'^2}\, dx \tag{3.6}$$

$$\Pi_4' = \frac{m_2 g}{l} \int_{x_1}^{x_2} x \sqrt{1 + y'^2}\, dx \tag{3.7}$$

Calculations deduce the equalities

$$\Pi_4' = (a^2 / 2) \sin \varphi \cos \varphi\, u_1 (z) \tag{3.8}$$

$$u_1 = (z \operatorname{ch} z - \operatorname{sh} z) / \operatorname{sh} z, \qquad z = (a \cos \varphi) / 2\alpha \tag{3.9}$$

$$\sqrt{l^2 - a^2 \sin^2 \varphi} / a \cos \varphi = \operatorname{sh} z / z \tag{3.10}$$

Formulas (3.9) indicate the form of the function $u_1 (z)$ and the relation between z and the quantities a, φ, α, while Eq. (3.10) defines the unique function $z (\sin \varphi, l / a) > 0$ increasing monotonically in both arguments in the domain D $[\sin \varphi \geqslant 0, l / a \geqslant 0]$ with observation of the relations

$$z (\sin \varphi \geqslant 0,\ l / a) \to \infty \quad \text{as} \quad l / a \to \infty \tag{3.11}$$

$$z\,(\sin\varphi,\ l\,/\,a > 1) \to \infty \quad \text{as} \quad \sin\varphi \to 1 \tag{3.12}$$

Denoting the derivatives with respect to φ of the sums $\Pi_3^{(1)} + \Pi_4$, $\Pi_3^{(2)} + \Pi_4$ by $\Pi_{(1)}'$, $\Pi_{(2)}'$, respectively, we obtain

$$\Pi_{(1)}' = gm_2 \sin\varphi\,[(a^2\,/\,2l)\,u_1(z)\cos\varphi - b\,(m_1+m_2)\,/\,m_2] \tag{3.13}$$

$$\Pi_{(2)}' = gm_2 \cos\varphi\,[(a^2\,/\,2l)u_1(z)\sin\varphi - b\,(m_1+m_2)\,/\,m_2] \tag{3.14}$$

Note that we need to set $-\sin\varphi$ in formula (3.8) in case $\sin\varphi < 0$, therefore, when $\varphi < 0$ we must change the sign of the first term with the brackets in formulas (3.13), (3.14).

3.1.1. We examine the details of the analysis of Case 3.1. In this case there exist two equilibria: an upper one ($\varphi = 0$) and a lower one ($\varphi = \pi$). The existence of a sloping ($\pi\,/\,2 > \varphi > 0$) equilibrium depends on the existence of a zero in the brackets. Replacing $\cos\varphi$ in the brackets from Eq. (3.10) in the form

$$\cos\varphi = \sqrt{l^2 - a^2}\,/\,a\sqrt{(\operatorname{sh} z\,/\,z)^2 - 1}$$

we obtain that the question of the existence of a sloping equilibrium is equivalent to the question of the existence of a positive solution of the equation

$$u_2(z) = zu_1(z)\,/\sqrt{\operatorname{sh}^2 z - z^2} - 2bl\,(m_1+m_2)\,/\,a\sqrt{l^2-a^2}\,m_2 = 0$$

Computing the derivative $u_2'(z)$, we obtain

$$u_2'(z) = \frac{z}{(\operatorname{sh}^2 z - z^2)^{3/2}}\left[\left(\frac{\operatorname{sh} z}{z} - \frac{z}{\operatorname{sh} z}\right)^2 - \left(\operatorname{ch} z - \frac{\operatorname{sh} z}{z}\right)^2\right]$$

The sign of the derivative $u_2'(z)$ is the same as the sign of the function

$$u_3(z) = 2\operatorname{sh} z\,/\,z - \operatorname{ch} z - z/\operatorname{sh} z$$

which admits of the representation in the form

$$u_3(z) = -(z\operatorname{sh} z)^{-1}\sum_{n=3}^{\infty}(2z)^{2n}\left(\frac{1}{(2n)!} - \frac{1}{4\,(2n-1)!}\right)$$

The last equality determines the estimate $u_2'(z) < 0$, which together with the equality

$$\lim\,(zu_1(z)\,/\sqrt{\operatorname{sh}^2 z - z^2}) = 0 \quad \text{as} \quad z \to \infty$$

enables us to make a deduction.

If $u_2\,(z\,(\sin\varphi = 0,\ l\,/\,a)) > 0$, then the upper ($\varphi = 0$) equilibrium position is stable, while the sloping ($\pi\,/\,2 < \varphi < 0$) one exists and corresponds to the maximum of the potential energy.

If $u_2\,(z\,(\sin\varphi = 0,\ l\,/\,a)) < 0$, then the upper ($\varphi = 0$) equilibrium position corresponds to the maximum of the potential energy, while the sloping equilibrium does not exist.

The lower ($\varphi = \pi$) equilibrium position is always stable. Two problems are of interest.

Problem 3.1.2. For a fixed filament mass and for a distance a between the fastening points, select the length l so as to maximize the restoring moment arizing under a deviation from the upper equilibrium.

Replacing l in the brackets in (3.13) by the formula $l = a\operatorname{sh} z\,/\,z$, we arrive at the problem of the maximum with respect to z of the function

$$u_4(z) = zu_1(z) / \operatorname{sh} z$$

The sign of the derivative $u_4'(z)$ is the same as the sign of the function

$$u_5(z) = z - 2 \operatorname{th} z / (2 - \operatorname{th}^2 z)$$

The function $u_5(z)$ decreases strictly from zero to the point $\operatorname{th} z = \sqrt{2/3}$, and then increases strictly to ∞. This means that the function $u_4(z)$ admits of a unique stationary maximum for $z > 0$ at the point

$$z^2{}_1 \approx 2.51, \qquad (l / a)_1 \approx 1.48$$

This equation determines the optimal length of the filament in the sense of Problem 3.1.2.

Problem 3.1.3. For fixed m_2, l, select the distance a which maximizes the restoring moment under a deviation from the upper equilibrium.

Problem 3.1.3 is equivalent to the problem of the maximum in the region $z > 0$ of the function $u_6(z) = z^2 u_1(z) / \operatorname{sh}^2 z$. The sign of the derivative $u_6'(z)$ is the same as the sign of the function

$$u_7(z) = \frac{z^2}{256}\left(-1 + \sum_{i=1}^{\infty} \alpha_i z^{2i}\right)$$

where within the parentheses there occurs a series with positive coefficients α_i, converging uniformly in any segment $|z| \leqslant N$ and tending to ∞ as $z \to \infty$. This also indicates the existence of a unique stationary maximum of the function $u_6(z)$ for $z_2^2 \approx 1.4$, $(l / a)_2 \approx 1.23$.

3.2.1. The upper ($\varphi = -\pi/2$) and lower ($\varphi = \pi / 2$) equilibrium positions are obvious for fastening 3.2, while the question of the existence of a sloping equilibrium reduces to the question of the existence of a positive root of the equation

$$u_8(z) = \sin\varphi \, u_1(z) - 2bl(m_1 + m_2) / a^2 m_2 = 0$$
$$\sin^2\varphi = 1 - (l^2 - a^2) / ((\operatorname{sh} z / z)^2 - 1) a^2$$

The function $u_8(z)$ grows monotonically in z from zero and tends to unity as $z \to \infty$ ($\varphi \to \pi / 2$). Denoting $v_1 = a^2 m_2 - 2bl(m_1 + m_2)$, we make some deductions.

Sloping equilibrium positions exist and are stable when $v_1 > 0$, while the upper and lower ones correspond to the maximum of $\Pi_{(2)}$. There are no sloping equilibria when $v_1 < 0$; the lower one is stable, and the upper one corresponds to the maximum of $\Pi_{(2)}$. It should be noted that in the upper and lower positions the filament hangs down along a straigh line which cannot be found directly from the filament's equilibrium conditions but only as the limit of its equilibria as $|x_1 - x_2| \to 0$.

If the body has been suspended from its center of gravity G_1, while the distance $G_1 O_1 = \lambda_3 (a / 2)$, $0 < \lambda_3 < 1$, then the condition $v_1 > 0$ takes the particularly simple form $a / l > \lambda_3$. The last estimate shows that the stable sloping equilibria of the filament in those cases when the filament is fastened at the ends A and B of a weightless rod and the rotation axis is located close to one of the ends, exist only for sufficiently short filaments.

4. An example of Problem 2.2 is a point mass m_1 sliding along the x-axis under the action of a force $F(\xi)$ (depending on the coordinate $\xi = x_2$ of mass m_1 and directed to the right, $F(\xi > 0) > 0$) and a filament of density $\mu_0 = m_2 / l$, fastened at points A $(x_1 = 0,\ y_1 = 0)$ and $B(x_2 = \xi > 0,\ y_2 = 0)$. It is assumed that the filament can slide in a plane P rotating with constant angular velocity ω around the x-axis and containing the y-axis.

By $\Pi_2(\xi)$ we denote the potential energy of the first halfwave and by $\Pi_2'(\xi)$ its derivative with respect to ξ. As a corollary of the equilibrium conditions we have the equations $\Pi_2'(\xi) = F(\xi) = (T\,dx/ds)_{x=\xi}$. The first equation expresses the condition of stationarity of the function $\Pi_1(\xi) + \Pi_2(\xi)$, while the second expresses the equality of the horizontal projection of the tension T at the right $(x = \xi)$ end of the filament and the force $F(\xi)$. On the other hand, it follows from the filament's equilibrium conditions [5] and from Eqs.(2.1) that the projection $T\,dx / ds$ is constant along the filament and

$$T\frac{dx}{ds} = \frac{\sqrt{2}}{8}\omega^2\frac{\mu_0\xi^2}{K^2k'^2}, \qquad k'K = \int_0^1 (1-k^2)^{1/2}((1-t^2)(1-k^2t^2))^{-1/2}\,dt$$

As we noted in Sect. 2, the quantity $k^2(\xi / l)$ decreases monotonically as ξ increases. Obviously, the product $k'K$ also decreases and, therefore, the variation of ξ from zero to l increases the detivarive $\Pi_2'(\xi)$ from zero up to the value

$$\Pi_2'(\xi = l) = v_2 = \sqrt{2}\,\omega^2 l^2\,\mu_0/2\pi^2$$

Assume that $F(\xi)$ is nonincreasing and continuous, then the condition $v_2 - F(l) > 0$ ensures the existence of a certain stable equilibrium $\xi_1 < l$ in which the filament has the form of a halfwave, while the equilibrium $\xi = l$ corresponds to the maximum of the potential energy $\Pi_1 + \Pi_2$. If, however, $v_2 - F(l) < 0$, the equilibrium $\xi = l$, in which the filament is rectilinear, becomes stable.

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